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Analytic continued fraction technique for bound and confined states for a class of confinement potentials

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Abstract. Analytic continued fraction theory is applied to study the convergence and analyticity of the infinite continued fraction representation of the Green function for a class of confinement potentials in terms of the Coulomb-like coupling constant. The possibility of a perturbative expansion in the powers (inverse powers) of the coupling constant is also investigated.

1. Introduction

The concepts of asymptotic freedom and quark confinement in non-Abelian gauge theories are expected to lead to a better understanding of the quark dynamics. The assumed high masses of the newly discovered flavours such as c and b permit one to consider the corresponding colour-singlet families as non-relativistic two-particle (quark–antiquark) bound states described by a Schrödinger equation (Appelquist and Politzer 1975). This picture is fairly successful in the charmonium model (Eichten *et al* 1976) description of the ψ family. The model assumes a potential consisting of a Coulombic term and a quark confining potential, whose form is not very precisely determined. It has been shown by Grosse (1977) and Martin (1977) that the correct order of levels can be obtained with any potential of the form $V_c(r) = br^{\sigma}$ where $0 < \sigma \leqslant 2$. A confining potential of the type

$$V(r) = -a/r + br + cr^2 \tag{1.1}$$

has also been studied (Gupta and Khare 1977) in this connection. The success of the charmonium model prompts one to treat the newly discovered Υ particles (Rosner *et al* 1978) in an exactly similar manner. But, the exact form of the quarkonium potential being unknown to a great extent, it is perhaps desirable to study the general analytic properties of a large class of confinement potentials. In this spirit, Singh *et al* (1977) and Khare (1978) studied the analyticity and eigenvalue problem of the Schrödinger equation with a class of potentials of the form (1.1). From the consistency requirement for the existence of the solutions of the corresponding difference equation, they constructed the Green function '$G(E)$' for the problem in the form of an infinite continued fraction and used the analytic theory of continued fractions to study its convergence and analyticity. They argued that the Green function can be transformed

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equivalently into an S-fraction of the form

$$\cfrac{1}{K_1\tilde{\alpha}+\cfrac{1}{K_2+\cfrac{1}{K_3\tilde{\alpha}+\cdots}}} \tag{1.2}$$

where the K_n are positive constants, the variable $\tilde{\alpha}$ being the inverse of the complexified harmonic coupling α, so that the theorems applicable to the S-fractions can be directly used to study the analyticity of the Green function. However, the quantities K_n are actually complicated functions of $\tilde{\alpha}$. We have shown later (first part of the proof of theorem 3.1) that the Green function does converge for positive real values of α, but the convergence for arbitrary α in the complex α-plane is not known. To overcome this difficulty, we have considered the convergence and analyticity of the Green function and its relation to the perturbation expansion in terms of the Coulomb coupling constant, instead of the harmonic coupling constant. This permits us to discuss the problem in a rigorous manner. The study is relevant also because, as pointed out by Eichten *et al* (1978), by varying the Coulomb parameter one can partially accommodate the short-distance effects such as (i) the spin-dependent forces and (ii) the logarithmic violation of scaling in the region of asymptotic freedom. Also, a perturbation expansion in the Coulomb parameter (or its inverse) may be used with advantage in many applications.

The presentation of the paper will be as follows. In § 2, we construct, following Singh *et al*, the Green function for the potential (1.1). In § 3, we discuss its convergence and analyticity and also the possibility of a perturbation expansion in the Coulomb parameter. In § 4, we specialise to the case of a harmonium potential ($b=0$). Our results are summarised in the last section.

2. Construction of the Green function

The radial Schrödinger equation in potential (1.1) is given by

$$R''(r)+[(2\mu/\hbar^2)(E+a/r-br-cr^2)-l(l+1)/r^2]R(r)=0 \tag{2.1}$$

where E is the energy of the system, and l the relative orbital angular momentum. We assume that $c>0$, while the signs of a and b are left free. The appropriate asymptotic behaviour of the solution of (2.1) is obtained by writing

$$R(r)=r^{l+1}\exp(-\tfrac{1}{2}\alpha r^2-\beta r)g(r) \tag{2.2}$$

where α, β are as yet unknown constants. The function $g(r)$ satisfies the equation

$$g''+2[(l+1)/r-\alpha r-\beta]g'+\{\epsilon-(2l+3)\alpha+[a-2\beta(l+1)]/r\}g=0 \tag{2.3}$$

where we choose

$$\alpha=+[(2\mu/\hbar^2)c]^{1/2}, \tag{2.4}$$

$$\beta=(2\mu/\hbar^2)^{1/2}b/c^{1/2}, \tag{2.5}$$

and write

$$\epsilon=\beta^2+e, \qquad e=(2\mu/\hbar^2)E. \tag{2.6}$$

To solve equation (2.3), we write

$$g(r)=\sum_{0}^{\infty} p_n r^n \tag{2.7}$$

and obtain the three-term difference equation

$$(n+2)(n+2l+3)p_{n+2}+[a+\gamma(n+l+2)]p_{n+1}+[\epsilon-(2n+2l+3)\alpha]p_n=0 \tag{2.8}$$

where $\gamma=-2\beta$. All p_n with negative values of n vanish identically with the choice $p_{-1}=0$. We can rewrite (2.8) as

$$\frac{p_{n+1}}{p_n}=\frac{-[\epsilon-(2n+2l+3)\alpha]}{\gamma(n+l+2)+a+(n+2)(n+2l+3)\dfrac{p_{n+2}}{p_{n+1}}}. \tag{2.9}$$

Using repeatedly equation (2.9) for $n=0, 1, 2, \ldots$, we obtain

$$\frac{p_1}{p_0}=\frac{-[\epsilon-(2l+3)\alpha]}{\gamma(l+2)+a-}\frac{2(2l+3)[\epsilon-(2l+5)\alpha]}{\gamma(l+3)+a-}\frac{3(2l+4)[\epsilon-(2l+7)\alpha]}{\gamma(l+4)+a-\cdots}. \tag{2.10}$$

Now, equating p_1/p_0 to $-[\gamma(1+l)+a]/(2l+2)$ which follows from (2.8) by putting $n=-1$, we obtain finally

$$\gamma(l+1)+a=\frac{(2l+2)[\epsilon-(2l+3)\alpha]}{\gamma(l+2)+a-}\frac{2(2l+3)[\epsilon-(2l+5)\alpha]}{\gamma(l+3)+a-\cdots}. \tag{2.11}$$

Equation (2.11) is the consistency condition for the existence of the solution for the system of equations (2.8). The solutions of (2.11) in the energy parameter are the energy eigenvalues for the problem, so that, following Singh *et al*, we define the 'Green function' of the problem as

$$G(\epsilon)=\frac{1}{b_1+\zeta-}\frac{a_1}{b_2+\zeta-}\frac{a_2}{b_3+\zeta-\cdots} \tag{2.12}$$

where

$$a_n=n(n+2l+1)[\epsilon-(2n+2l+1)\alpha],$$
$$b_n=\gamma(n+l), \qquad n=1,2,3,\ldots. \tag{2.13}$$

We replaced a, the Coulomb parameter, by a complex variable ζ with respect to which we shall discuss the analyticity of the Green function (2.12). The Green function, by construction, has a pole whenever E equals the binding energy of a bound or confined state with a real ζ. We note that the infinite continued fraction (2.12) representing the Green function is a J-function in ζ. In appendix 1 we shall show that this infinite continued fraction representation of the Green function converges to a real meromorphic function in the energy parameter ϵ.

3. Convergence and analyticity

3.1. Green's function

We now discuss the convergence and analyticity of the J-fraction representation (2.12) of the Green function. We first consider the case for the bound-state problem and assume that $\epsilon < 0$ and write $\epsilon = -\varepsilon$, $\varepsilon > 0$. We also choose the linear confinement coupling $b < 0$. The harmonic coupling c is always positive. We then rewrite (2.12) and (2.13):

$$G(\epsilon) = \frac{1}{b_1 + \zeta +} \frac{a_1}{b_2 + \zeta +} \frac{a_2}{b_3 + \zeta + \cdots} \tag{3.1}$$

and

$$a_n = n(n+2l+1)[\varepsilon + (2n+2l+1)\alpha] > 0,$$
$$b_n = \gamma(n+l) > 0, \qquad n = 1, 2, 3, \ldots. \tag{3.2}$$

We derive the convergence and analyticity of the continued fraction (3.1), using a convergence theorem due to Van Vleck (Wall 1948, theorem 30.1, p 131). We first obtain the following results.

3.1.1. Under the equivalence transformation

$$c_1 = 1, \qquad c_{n+1}c_n a_n = 1, \qquad c_n(b_n + \zeta) = d_n, \tag{3.3}$$

the continued fraction (3.1) takes the form

$$G(\epsilon) = \frac{1}{d_1 +} \frac{1}{d_2 +} \frac{1}{d_3 + \cdots} \tag{3.4}$$

where $d_1 = b_1 + \zeta$, $\zeta = x + \mathrm{i}y$. For the applicability of Van Vleck's theorem, the partial denominators of (3.4) should satisfy the inequalities

$$\mathrm{Re}(d_1) > 0, \qquad |\mathrm{Im}(d_n)| \leq r\, \mathrm{Re}(d_n), \qquad n = 1, 2, 3, \ldots, \tag{3.5}$$

with r any positive number. Clearly, these inequalities define a domain S in the ζ plane

$$S = \{\zeta = x + \mathrm{i}y : |y| \leq r(b_1 + x),\ r > 0\}. \tag{3.6}$$

Since $r > 0$ is arbitrary, S actually constitutes the half plane $\mathrm{Re}(\zeta) \geq -b_1$. Also $\mathrm{Re}(d_n) > 0$, as can be seen from (3.2) and (3.3).

3.1.2. Consider the set of transformations

$$t_n(\omega) = 1/(d_n + \omega) \tag{3.7}$$

generating the continued fraction (3.4) by the definition

$$G(\epsilon) = \lim_{n\to\infty} t_1 t_2 \ldots t_n(0).$$

Since $\mathrm{Re}(d_n) > 0$ for each n, it is clear that $t_n = t_n(\omega)$ maps the right half-plane $\mathrm{Re}(\omega) \geq 0$ into the right half-plane $\mathrm{Re}(t_n) \geq 0$. In particular, $t_1 = t_1(\omega)$ maps $\mathrm{Re}(\omega) \geq 0$ into the circular domain

$$|t_1 - 1/2\,\mathrm{Re}(d_1)| \leqslant 1/2\,\mathrm{Re}(d_1). \tag{3.8}$$

We easily see that the transformation $T_n = t_1 t_2 \ldots t_n(\omega)$ also maps $\mathrm{Re}(\omega) \geqslant 0$ into the domain (3.8). Now, since $t_1 t_2 \ldots t_n(0) = A_n(\zeta)/B_n(\zeta)$, the nth approximant of the continued fraction, we obtain

$$\begin{aligned}|A_n(\zeta)/B_n(\zeta)| &\leqslant 1/\mathrm{Re}\, d_1 \\ &\leqslant 1/\delta, \qquad \mathrm{Re}(\zeta) \geqslant -b_1 + \delta > -b_1\end{aligned} \tag{3.9}$$

for each n. Hence, the sequence of approximants is uniformly bounded over every finite closed domain in the half plane $\mathrm{Re}(\zeta) > -b_1$.

3.1.3. It is easily seen from the equations (3.3) that

$$\begin{aligned}c_{2n+1} = {} & \frac{\Gamma(n+1/2)(n+l)!}{\Gamma(1/2)n!l!} \frac{\Gamma(l+3/2)}{\Gamma(n+l+3/2)} \\ & \times \frac{\Gamma[n+\frac{1}{4}(2l+3+\varepsilon/\alpha)]}{\Gamma[\frac{1}{4}(2l+3+\varepsilon/\alpha)]} \frac{\Gamma[\frac{1}{4}(2l+5+\varepsilon/\alpha)]}{\Gamma[n+\frac{1}{4}(2l+5+\varepsilon/\alpha)]},\end{aligned} \tag{3.10}$$

$$\begin{aligned}c_{2n+2} = {} & \frac{n!l!\Gamma(1/2)}{2^4\alpha\Gamma(n+3/2)(n+l+1)!} \frac{\Gamma(n+l+3/2)}{\Gamma(l+3/2)} \\ & \times \frac{\Gamma[n+\frac{1}{4}(2l+5+\varepsilon/\alpha)]}{\Gamma[\frac{1}{4}(2l+5+\varepsilon/\alpha)]} \frac{\Gamma[\frac{1}{4}(2l+3+\varepsilon/\alpha)]}{\Gamma[n+\frac{1}{4}(2l+7+\varepsilon/\alpha)]}.\end{aligned} \tag{3.11}$$

Since for large n, c_n behaves as $n^{-3/2}$, we see that $|d_n| \sim n^{-1/2}$ and thus $\Sigma\,|d_n|$ is divergent. We now prove the following theorem.

Theorem 3.1. The continued fraction (3.1) for $\varepsilon > 0$ is uniformly convergent over every finite closed domain in the half plane $\mathrm{Re}(\zeta) > -b_1$ and its value is an analytic function in that half plane.

Proof. Let ζ be real and $\zeta > -b_1$. Then it follows immediately from Van Vleck's theorem and § 3.1.3 that the continued fraction (3.4), and hence the continued fraction (3.1), is convergent. Now, if S' is any bounded closed domain entirely within the half plane $\mathrm{Re}(\zeta) > -b_1$, it follows from (3.9) that the approximants of the continued fraction (3.1) are uniformly bounded over S'. Hence, by the convergence continuation theorem (Wall 1948, theorem 24.2, p 108) we obtain the theorem.

The continued fraction (3.1) is also uniformly convergent to an analytic function in the left half-plane $\mathrm{Re}(\zeta) < -b_1$, which can be seen from the following reflection argument.

We write $z = \zeta + b_1$ in (3.1), so that

$$G(\epsilon) = \cfrac{1}{z + \cfrac{a_1}{\gamma + z + \cfrac{a_2}{2\gamma + z + \cdot}}}.$$

which converges to an analytic function for $\mathrm{Re}(z) > 0$. We now replace z by $-z$, and

$G(\epsilon)$ in the half plane $\mathrm{Re}(z)<0$ takes the form (after an equivalence transformation)

$$G(\epsilon)=\frac{-1}{z+}\frac{a_1}{-\gamma+z+}\frac{a_2}{-2\gamma+z+\cdots}.$$

This continued fraction is clearly convergent to an analytic function for $\mathrm{Re}(z)<0$. Thus $G(\epsilon)$ is also analytic in the half plane $\mathrm{Re}(\zeta)<-b_1$. Moreover, the two analytic functions in the half planes $\mathrm{Re}(\zeta)\gtrless -b_1$ are actually analytic continuations of each other. This is easily seen if we consider values of ζ such that the partial denominators of (3.4) satisfy the inequalities

$$\mathrm{Re}(d_1)<0, \qquad \mathrm{Re}(d_2)>0, \qquad |\mathrm{Im}(d_n)|\leq r\,\mathrm{Re}(d_n), \qquad r>0;$$
$$n=2,3,\ldots.$$

In that case theorem 3.1 can be applied to the continued fraction

$$\frac{1}{d_2+}\frac{1}{d_3+\cdots}. \tag{3.12}$$

Thus the continued fraction (3.12) converges uniformly to an analytic function $\gamma(\zeta)$ in the half plane $\mathrm{Re}(\zeta)>-b_2$, and hence

$$G(\epsilon)=1/[b_1+\zeta+f(\zeta)]$$

is meromorphic in $\mathrm{Re}(\zeta)>-b_2$ ($b_2>b_1$).

We therefore conclude that the two analytic functions in the half planes $\mathrm{Re}(\zeta)\gtrless -b_1$ actually constitute a single meromorphic function in the whole ζ plane whose poles lie on the line $\mathrm{Re}(\zeta)=-b_1$. We state the above result in the following theorem.

Theorem 3.2. The *J*-fraction representation (3.1) of the Green function for $\epsilon<0$ for the potential

$$V(r)=-a/r+br+cr^2, \qquad b<0, c>0,$$

converges to a meromorphic function of ζ whose poles lie only on the line $\mathrm{Re}(\zeta)=-b_1$. The convergence is uniform over every finite closed domain containing none of the poles of this function.

To study the convergence of the Green function in the case of the confined states, we note that in such cases we must have $\epsilon>0$. It should be noted that $\epsilon>0$ ($\epsilon=\beta^2+e$) also includes bound states with energy $-\beta^2<e<0$.

For $\epsilon>0$, there exists a positive integer m such that

$$a_n>0,\ n<m, \qquad a_n\leq 0,\ n\geq m,$$

holds and we can write (2.12) in the form

$$G(\epsilon)=\cfrac{1}{b_1+\zeta-\cfrac{a_2}{b_2+\zeta-\ddots-\cfrac{a_{m-2}}{b_{m-1}+\zeta-a_{m-1}G_m}}}, \tag{3.13}$$

$$G_m = \cfrac{1}{b_m + \zeta - \cfrac{a_m}{b_{m+1} + \zeta - \ddots}} \tag{3.14}$$

Theorem 3.2 holds for G_m; let it converge to a meromorphic function $g_m(\zeta)$. The total Green function $G(\epsilon)$ can be written as

$$\frac{A_{m-1}(\zeta) - a_{m-1} g_m(\zeta) A_{m-2}(\zeta)}{B_{m-1}(\zeta) - a_{m-1} g_m(\zeta) B_{m-2}(\zeta)}$$

where the denominator does not vanish identically. The reason for this is that the two functions $g_m(\zeta)$ and $B_{m-1}(\zeta)/B_{m-2}(\zeta)$ possess different singularity structures; the poles of the former lie on the line $\mathrm{Re}(\zeta) = -b_m$ whereas that of the latter are all real, $B_{m-2}(\zeta)$ being the denominator of a (finite) real J-fraction (Wall 1948, theorem 27.1, p 114). Thus we have the following corollary.

Corollary 3.3. The Green function $G(\epsilon)$ for $\epsilon > 0$ is a meromorphic function in ζ, real poles of which correspond to the real eigenvalue problem.

To study further analytic properties of the Green function, we next write

$$\zeta = 1/z$$

in the continued fraction (2.12) to reduce it to the form

$$G = zF \tag{3.15}$$

where

$$F = \cfrac{1}{b_1 z + 1 - \cfrac{a_1 z^2}{b_2 z + 1 - \ddots}}$$

The J-fraction F converges to a meromorphic function of z for all values of z, with a possible exception at $z = 0$. But $F(0) = 1$, and this implies that F converges in the neighbourhood of the origin of the z plane, $|z| < \delta$ where δ is the distance of the nearest pole. Thus the power-series expansion of the J-fraction F, $\Sigma_0^\infty \omega_r z^r$, has radius of convergence at least equal to δ and its value is equal to that of the J-fraction (Wall 1948, theorem 54.1, p 208). Since the corresponding results hold for G we conclude the following.

Theorem 3.4. The perturbation series for the Green function in the inverse powers of the Coulomb coupling (ζ) converges for sufficie ıtly large values of ζ.

3.2. Energy eigenvalues

The Green function $G(\epsilon)$ is a meromorphic function of ζ for any fixed real γ and ϵ. Now, for $\zeta = 0$, the eigenvalue problem reduces to the corresponding problem for the potential

$$V(r) = br + cr^2, \tag{3.16}$$

and hence the consistency condition at $\zeta = 0$

$$G^{-1}(\epsilon) = 0$$

has an infinite set of solutions $\epsilon = \epsilon_0^n$ for a fixed $\gamma = \bar{\gamma}$ (say). Thus $G(\zeta, \epsilon, \bar{\gamma})$ is analytic in ζ for any fixed real ϵ except the infinite set of eigenvalues $\epsilon = \epsilon_0^n$ at which G has a pole at $\zeta = 0$. Hence $G^{-1}(\zeta, \epsilon)$ is analytic at $\zeta = 0$, for any fixed real ϵ in the neighbourhood of an eigenvalue $\epsilon = \epsilon_0$.

Now, for the analyticity of $G(\zeta, \epsilon)$ in ϵ we can easily see that G converges uniformly to a real meromorphic function in ϵ for any fixed real ζ (see appendix 1). Hence $G^{-1}(\zeta, \epsilon)$ is real analytic in ϵ in the neighbourhood of an eigenvalue $\epsilon = \epsilon_0$, for any fixed real ζ near $\zeta = 0$.

Thus we have proved that $G^{-1}(\zeta, \epsilon)$, as a function of two variables, is analytic in ζ in the neighbourhood of $\zeta = 0$, for any fixed real ϵ in the neighbourhood of $\epsilon = \epsilon_0$, and also analytic in ϵ in the neighbourhood of $\epsilon = \epsilon_0$, for any fixed real ζ in the neighbourhood of $\zeta = 0$. Hence by the Malgrange–Zerner theorem (Epstein 1966) (see appendix 2) we see that $G^{-1}(\zeta, \epsilon)$ is analytic in the two variables in the neighbourhood of $\zeta = 0$, $\epsilon = \epsilon_0$.

Moreover, $G(0, \epsilon)$ can have only simple poles in the ϵ plane, since each energy eigenvalue is non-degenerate in this problem. This shows that $G^{-1}(0, \epsilon)$ and $(d/d\epsilon)G^{-1}(0, \epsilon)$ cannot both vanish at the same point. Thus, at the eigenvalue $\epsilon = \epsilon_0$,

$$G^{-1}(0, \epsilon_0) = 0, \qquad (d/d\epsilon)G^{-1}(0, \epsilon_0) \neq 0.$$

Hence by the implicit function theorem $G^{-1}(\zeta, \epsilon) = 0$ has a unique solution $\epsilon(\zeta)$ which is analytic in ζ at $\zeta = 0$.

We have thus proved the following.

Theorem 3.5. The eigenvalue $\epsilon(\zeta)$ is analytic at $\zeta = 0$, and hence the formal perturbation series of $\epsilon(\zeta)$ in the powers of the coupling constant is convergent.

4. Harmonium potential

We now specialise the potential (1.1) to the harmonium potential ($b = 0$). The Green function (3.1) then reduces to the simpler form

$$G(e) = \cfrac{1}{\zeta + \cfrac{a_1}{\zeta + \cfrac{a_2}{\zeta + \cdots}}} \tag{4.1}$$

The J-fraction (4.1) is equivalent to an S-fraction. For, under two successive equivalence transformations

$$c_{n+1}c_n a_n = \tilde{a}_n, \qquad c_{2n+1} = \zeta, \qquad c_{2n} = \zeta^{-1}, \tag{4.2a}$$

and

$$\rho_1 = 1, \qquad \rho_{n+1}\rho_n\tilde{a}_n = 1, \qquad n = 1, 2, 3, \ldots, \tag{4.2b}$$

the equation (4.1) becomes

$$\frac{1}{\sqrt{z}} G(e) = \cfrac{1}{\rho_1 z + \cfrac{1}{\rho_2 + \cfrac{1}{\rho_3 z + \cdots}}} \tag{4.3}$$

where we write $z = \zeta^2$. The continued fraction (4.2) is a standard S-fraction. Since we have assumed the bound-state condition $e < 0$, the a_n's are positive and hence the ρ_n's are positive constants. We also note that $\Sigma\, \rho_n$ is convergent.

Hence we obtain (Wall 1948, theorem 28.2, p 120) the following results. The S-fraction representation of the Green function for the harmonium potential diverges by oscillation. The even and odd parts converge to distinct meromorphic functions uniformly over every finite closed domain whose distance from the negative half of the real axis is positive. The poles of the limit functions are all on the negative real axis of the ζ^2 plane.

But clearly $\zeta = 0$ is a square-root branch point. Choosing the branch of the first Riemann sheet, we conclude that the two distinct limit functions are actually analytic in the ζ plane cut along the positive imaginary axis. Hence there exists no eigenvalue for the harmonium potential under the bound-state condition $e < 0$.

But for $e > 0$, eigenvalues corresponding to the confined states exist. This follows from the analysis applied to the continued fractions (3.13) and (3.14) with $b_n = 0$ ($n = 1, 2, 3, \ldots$). We obtain two meromorphic functions in the cut ζ-plane representing the Green function. The real singularities of these functions actually correspond to the eigenvalue problem.

Finally, since the analyticity of the Green function (in fact, of its parts) can only be obtained in the ζ plane with a cut running from $\zeta = 0$ to $\zeta = \mathrm{i}\infty$, the perturbative expansion in ζ (or in ζ^{-1}) must have a radius of convergence equal to zero, which follows directly from the arguments of § 3.2. Thus for the harmonium potential, we obtain the following theorems.

Theorem 4.1. The perturbation series for the Green function of the harmonium potential in the inverse powers of the Coulomb parameter is divergent.

Theorem 4.2. The energy perturbation series in the Coulomb parameter is divergent.

5. Summary

We have studied the analyticity of the infinite continued fraction representation of the Green functions in the Coulomb coupling constant (ζ) for the confinement potentials:

(i) a combination of harmonium and linear potentials;

(ii) a pure harmonium potential.

In the case (i), it is found that the Green function converges to a meromorphic function uniformly over every finite closed domain containing none of the poles of the function. The real poles of this function correspond to the real eigenvalue problem. The analyticity also ensures the convergence of the energy perturbation series in ζ. But in the case (ii), the Green function diverges by oscillation for every value of ζ^2. Still, the even and odd parts of the J-fraction representing the Green function possess definite analytic properties in the ζ plane cut along the positive imaginary axis, thus furnishing a solution to the eigenvalue problem. An interesting conclusion is that the energy levels for the harmonium potential are all positive (confined). Moreover, the divergence of the Green function, and also the analyticity of its parts only in a cut plane, render the perturbation expansion in ζ (or in ζ^{-1}) totally divergent. Finally, it should be noted that the continued fraction representation, in any case, affords one with an alternative but rapid method of computing the energy levels.

Appendix 1. Analyticity of $G(\epsilon)$ for real ϵ

Case 1. Let $\epsilon = -\varepsilon$, $\varepsilon > 0$; $\zeta \geqslant -b_1$ and let ζ be fixed. Then by the parabola theorem (Wall 1948, theorem 14.2, pp 59–60) $G(\epsilon)$ given by (3.1) is convergent for each $\varepsilon > 0$ and the approximants lie in the interval

$$|x-1| \leqslant 1, \qquad x \neq 0.$$

Hence, by the convergence continuation theorem, $G(\epsilon)$ converges uniformly to an analytic function of $\varepsilon\,(0 \leqslant \varepsilon < \infty)$. For fixed $\zeta < -b_1$, the reflection argument used in the paragraph below theorem 3.1 is applicable, and hence $G(\epsilon)$ converges to an analytic function of ε for any fixed real ζ.

Case 2. $\epsilon > 0$. Arguments for corollary 3.3 are applicable in this case. We note, in particular, that the denominator

$$B_{m-1}(\epsilon) - a_{m-1} g_m(\epsilon) B_{m-2}(\epsilon)$$

does not vanish identically for $0 \leqslant \epsilon < \infty$ since $g_m(\epsilon)$ is analytic in ϵ for any fixed ζ, whereas $B_{m-1}(\epsilon)/a_{m-1}B_{m-2}(\epsilon)$ may have a pole for a special choice of fixed real ζ.

Hence $G(\epsilon)$ converges to a real meromorphic function of $\epsilon\,(-\infty < \epsilon < \infty)$.

Appendix 2.

We prove here the analyticity of $G^{-1}(\zeta, \epsilon)$ in two variables by writing

$$F(\zeta, \epsilon) = G^{-1}(\zeta, \epsilon), \qquad \zeta = \zeta_1 + \mathrm{i}\zeta_2, \qquad \epsilon = \epsilon_1 + \mathrm{i}\epsilon_2,$$

$$f_1(\zeta, \epsilon_1) = F(\zeta, \epsilon_1), \qquad f_2(\zeta_1, \epsilon) = F(\zeta_1, \epsilon).$$

Then (i) $f_1(\zeta, \epsilon_1)$ is analytic in ζ in the neighbourhood of $\zeta = 0$; (ii) $f_2(\zeta_1, \epsilon)$ is analytic in ϵ in the neighbourhood of $\epsilon = \epsilon_0$.

Thus f_1 and f_2 have the properties stated in the Malgrange–Zerner theorem (Epstein 1966). Hence there exists a function $\tilde{F}(\zeta, \epsilon)$ analytic in (ζ, ϵ) in the domain

$$\{(\zeta, \epsilon): (\zeta_1, \epsilon_1) \text{ in the neighbourhood of } (0, \epsilon_0);\ 0 \leqslant \zeta_2 < \delta_1,$$
$$0 \leqslant \epsilon_2 < \delta_2,\ \delta_1 < 1,\ \delta_2 < 1;\ \zeta_2 + \epsilon_2 < 1\}.$$

Hence, by continuity, $\tilde{F}(\zeta, \epsilon)$ is analytic in the neighbourhood of $(0, \epsilon_0)$ and $\tilde{F}(\zeta, \epsilon) = F(\zeta, \epsilon)$.

References

Appelquist T and Politzer H D 1975 *Phys. Rev. Lett.* **34** 43–5
Eichten E, Gottfried K, Kinoshita T, Lane K D and Yan T M 1976 *Phys. Rev. Lett.* **36** 500–4
—— 1978 *Phys. Rev.* D **17** 3090–117
Epstein H 1966 *Axiomatic Field Theory* ed M Chretian and S Deser (New York: Gordon and Breach)
Grosse H 1977 *Phys. Lett.* **68B** 343–6
Gupta V and Khare A 1977 *Phys. Lett.* **70B** 313–6
Khare A 1978 *Preprint* Institute of Physics, Bhubaneswar IP/BBSR/78–18
Martin A 1977 *Phys. Lett.* **67B** 330–4
Rosner J L, Quigg C and Thacker H B 1978 *Phys. Lett.* **74B** 350–2
Singh V, Biswas S N and Datta K 1977 *Preprint* Delhi University
Wall H S 1948 *Analytic Theory of Continued Fractions* (New York: Van Nostrand)